2019-Oct-01 (lecture) MATB44-2019 Prof. Yun William Yu

Tuesday, October 1, 2019 3:05 PM

Last time:

- We proved the Banach fixed pt Theorem
- We covered Picard iteration for vectors

This time:

- Proof of Picard-Lindelöf existence-uniqueness thm
- Examples applying Picard-Lindelöf
- Review complex numbers

Theorem (Picard-Lindelöf):
(Tesch 2.2)

Suppose $f: C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. If f is a locally Lipschitz continuous in the 2nd argument, uniformly with respect to the 1st argument, then there exits a unique local solution $\bar{x}(t) \in C^1(I)$ of the initial value problem, where I is some closed interval around t_0. ← mistake in old notes

Proof.

Let's say $t_0 = 0$ for notational simplicity. i.e. IVP $x(0) = x_0$.
In order to apply the Banach fixed pt thm, we need a Banach space.

Let's try $X = C([0,T], \mathbb{R}^n)$ for suitable $T > 0$.
and $\|x\| = \sup_{t \in [0,T]} \|x(t)\|$, where we take the Euclidean norm in $\mathbb{R}^n$. (rather than abs value in $\mathbb{R}$)

Because $U \subseteq \mathbb{R}^{n+1}$ is open, we can choose to work under a closed ball of radius δ around x_0.

i.e. Let $V = [0,T] \times \overline{B_\delta(x_0)} \subset U$

where $\overline{B_\delta(x_0)} = \{ x \in \mathbb{R}^n \mid \|x - x_0\| \le \delta \}$.

Let $M = \max\limits_{(t,x)\in V} \|f(t,x)\|$, which exists by continuity of f and compactness of V.

Now, let's recall the Picard iteration mapping

$$K(x)(t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

Then $\|K(x)(t) - x_0\| \le \int_0^t \|f(s,x(s))\|\, ds \le tM$ when the graph $x(t)$ lies in V.

But $x(t)$ might escape $\overline{B_\delta(x_0)}$ if $tM > \delta$.

So let $T_0 = \min\{T, \frac{\delta}{M}\}$, ensuring that on

$$V_0 = [0, T_0] \times \overline{B_\delta(x_0)} \subseteq V,$$

$$\|K(x)(t) - x_0\| \le \int_0^t \|f(s, x(s))\| ds \le tM.$$

So, let's try $X = C([0,T_0], \mathbb{R}^n)$ as our Banach space, with norm $\|x\| = \sup\limits_{t\in[0,T_0]} \|x(t)\|$, and

Let $C = \{ x \in X \mid \|x - x_0\| \le \delta \}$ as our closed subset, the set of continuous functions on $[0, T_0]$ that remain within distance δ of x_0.

We chose T_0 so that $\int_0^t \|f(s,x(s))\| dt \le tM \le T_0 M \le \delta$, so

$$K(x)(t) = x_0 + \int^t f(s, x(s)) \in C$$

$$K(x)(t) = x_0 + \int_0^t f(s, x(s)) \in C$$

Thus $K: C \to C$.

Now we need to show that K is a contraction.

Recall that we assumed f is locally Lipschitz continuous in the 2nd argument, uniformly with respect to the first.

For simplicity, we are going to assume that f is Lipschitz continuous on V_0.

For all (t_0, x), $(t_1, y) \in V_0$, there exists a finite $L \geq 0$ s.t.

$$\| f(t_0, x) - f(t_1, y) \| \leq L \| (t_0, x) - (t_1, y) \|.$$

Let $t_0 = t_1 = t$.

Then $\forall\, (t,x), (t,y) \in V_0$, there exists a finite $L \geq 0$ s.t.

$$\| f(t,x) - f(t,y) \| \leq L \| (t,x) - (t,y) \| = L \| x - y \|.$$

This implies that we can choose

$$L = \sup_{(t,x) \neq (t,y) \in V_0} \frac{\| f(t,x) - f(t,y) \|}{\| x - y \|}.$$

Then
$$\int_0^t \| f(s, x(s)) - f(s, y(s)) \| ds \leq L \int_0^t \| x(s) - y(s) \| ds$$
$$\leq L\, t \sup_{0 \leq s \leq t} \| x(s) - y(s) \|,$$

so long as both $x(t)$ and $y(t)$ lie in V_0.

But
$$\| K(x) - K(y) \| = \left\| \left(x_0 + \int_0^t f(s, x(s))\, ds \right) - \left(x_0 + \int_0^t f(s, y(s)) ds \right) \right\|$$
$$= \left\| \int_0^t (f(s, x(s)) - f(s, y(s)))\, ds \right\|$$

$$= \left\| \int_0^t \left(f(s, x(s)) - f(s, y(s))\right) ds \right\|$$

$$\leq \int_0^t \| f(s, x(s)) - f(s, y(s)) \| \, ds$$

And $L t \sup_{0 \leq s \leq t} \| x(s) - y(s) \| \leq L T_0 \| x - y \|$.

$$\Rightarrow \quad \| K(x) - K(y) \| \leq L \, T_0 \, \| x - y \|.$$

Remember that we chose $T_0 = \min \{ T, \frac{\delta}{M} \}$.

Let's now make T_0 even smaller, choosing $T_0 < L^{-1}$.

Then K is a contraction on C, proving the claim. $\square$

We have now proven that there exists a unique local solution to the IVP for a system of first-order ODEs, so long as $\dot{x} = f(t, x)$ has $f(t, x)$ be locally Lipschitz continuous.

Theorem: Improved Picard-Lindelöf
(Teschl 2.5)

Suppose $f \in C(U, \mathbb{R}^n)$, where $U \subseteq \mathbb{R}^{n+1}$ an open subset, and f is locally Lipschitz continuous in the 2nd argument.
Choose $(t_0, x_0) \in U$, and $\delta, T > 0$ s.t. $[t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$.

Set
$$M(t) = \int_{t_0}^{t} \sup_{x \in B_\delta(x_0)} | f(s, x) | \, ds.$$

$$L(t) = \sup_{x \neq y \in B_\delta(x_0)} \frac{| f(t, x) - f(t, y) |}{| x - y |}.$$

$$T_0 = \sup \{ 0 < t \leq T \mid M(t_0 + t) \leq \delta \}.$$

Suppose
$$\int_{t_0}^{t_0 + T_0} L(t) \, dt < \infty$$

Then the unique local solution $\bar{x}(t)$ of the IVP is given by
$$\bar{x} = \lim_{m \to \infty} K^m(x_0) \in C^1([t_0, t_0 + T_0], \overline{B_\delta(x_0)}),$$

by $$\bar{x} = \lim_{m\to\infty} K^m(x_0) \in C^1([t_0, t_0+T_0], \overline{B_\delta(x_0)}),$$
where K is Picard iteration.

Note that we only require that f is locally Lipschitz continuous in the 2nd argument, but have to have
$$\int_{t_0}^{t_0+T_0} L(t)\,dt < \infty$$

Examples: $\dot{x} = x^2 + t$, $x(0) = 1$.

Is $f(t,x) = x^2 + t$ locally Lipschitz continuous in the 2nd arg?

Need $|f(t,x) - f(t,y)| < L|x-y|$ for some finite L in a neighborhood of $(0,1)$, $x \neq y$.

Consider $V_0 = [-1,1] \times [0,2]$

Then $$\frac{|f(t,x) - f(t,y)|}{|x-y|} = \frac{|x^2-y^2|}{|x-y|} = \frac{|x-y||x+y|}{|x-y|} = |x+y| < 4.$$

Thus, we have that in V_0, $f(x,t) = x^2+t$ is locally Lipschitz continuous in the 2nd arg, uniformly w.r.t. to the first arg.

This implies by Picard-Lindelöf that there exists a unique local solution to $\dot{x} = x^2+t$ around $(0,1)$.

Ex. $\dot{x} = t^2x^2$, $x(5) = 10$.

Consider $V_0 = \underset{t}{[0,10]} \times \underset{x}{[0,20]}$, let $f(t,x) = t^2x^2$

Then $$\frac{|f(t,x) - f(t,y)|}{|x-y|} = \frac{|t^2x^2 - t^2y^2|}{|x-y|} = \frac{t^2|x^2-y^2|}{|x-y|} = t^2|x+y|$$
$$\leq 100 \cdot 40 = 4000 < \infty$$

Thus, by Picard-Lindelöf, we have a local unique solution.

Ex. $\dot{x} = \sqrt{|x|}$, $x(0) = 0$.

Consider $V_0 = [-1,1] \times [-1,1]$.

Then $$\frac{\left|\sqrt{|x|} - \sqrt{|y|}\right|}{|x-y|} = \frac{\left|\sqrt{|x|} - \sqrt{|y|}\right|}{\left|\sqrt{|x|} - \sqrt{|y|}\right|\left|\sqrt{|x|} + \sqrt{|y|}\right|} = \frac{1}{\left|\sqrt{|x|} + \sqrt{|y|}\right|} \to \infty \text{ if } x, y \to 0$$

Thus, we cannot show that there exists a unique local solution

(In fact there isn't. See Tesch 1.3)

Recall: Having a continuous derivative implies local Lipschitz continuity on a compact set.

(Teschl problem 2.5)

Ex $f(t,x) = t^2 + x^2$.

Note $\frac{\partial f}{\partial t} = 2t$ and $\frac{\partial f}{\partial x} = 2x$ are continuous, so

$f \in C^1(\mathbb{R}^2, \mathbb{R})$, i.e. f has a continuous total derivative

So f is locally Lipschitz continuous everywhere.

Thus $\dot{x} = f(t,x)$ has a unique local solution around every point (t, x_0).

Note: If $f(t,x) = A(t)x, + b(t)$ where $A(t)$ is a matrix and $b(t)$ a vector, and both have continuous derivatives, then there exists a unique global solution.

Complex numbers: Consider the set of complex numbers defined by $z = x + yi$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

We call yi, $y \in \mathbb{R}$ to be an imaginary number.

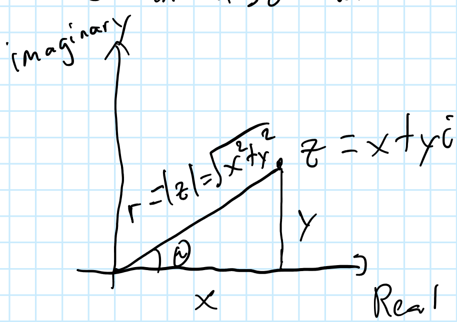
Given $z = x + yi$, the imaginary part $\operatorname{Im}(z) = y$.

the real part $\operatorname{Re}(z) = x$,

the complex conjugate $\bar{z} = x - yi$

the modulus of z $\quad |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$

We can also write complex numbers in polar form (as opposed to rectangular)



Let $r = |z| = \sqrt{x^2 + y^2}$

Then $x = r\cos\theta$

$y = r\sin\theta$

We say that $\theta = \operatorname{Arg} z$ is the smallest positive angle satisfying $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$.

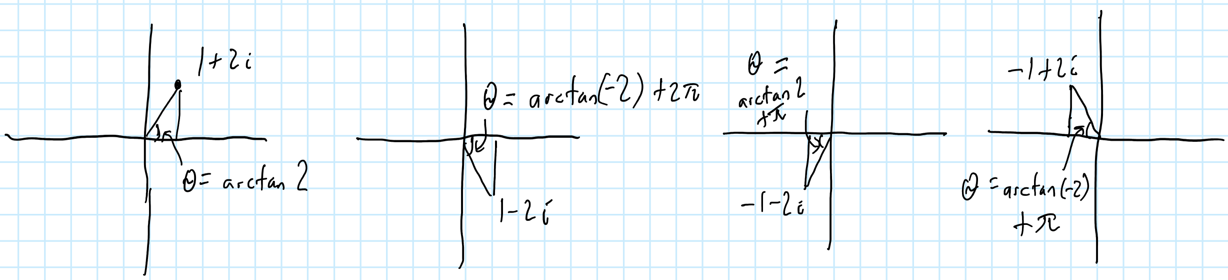
When $x, y > 0$, $\quad \theta = \arctan\frac{y}{x}$

$x > 0,\ y \le 0$, $\quad \theta = \arctan\frac{y}{x} + 2\pi$

$x < 0,\ y \ge 0$, $\quad \theta = \arctan\frac{y}{x} + \pi$

$x < 0,\ y < 0$, $\quad \theta = \arctan\frac{y}{x} + \pi$

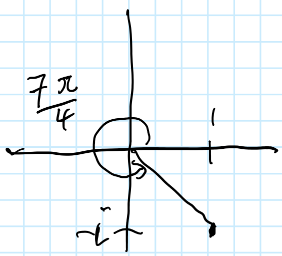
However, instead of memorizing, it is often easier to draw out the appropriate triangles on the complex plain:



Note that we define $\arg z = \operatorname{Arg} z \pm 2n\pi$, $\quad n \in \mathbb{Z}$.

$z = r(\cos\theta + i\sin\theta)$ is the polar form.

Ex. Say $z = 1 - i$ (rectangular form)



$$|z| = \sqrt{2}, \quad \operatorname{Arg} z = \frac{7\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right).$$ (polar form)

Claim: $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Proof. Write in polar form $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\right]$$

$$= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$\square$

Mentimeter:

$$|1| = 1$$

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$|2-2i| = \sqrt{2^2+2^2} = \sqrt{8}$$

$$|3+i| = \sqrt{9+1} = \sqrt{10}$$

Definitions: Let's define e^z

Use a power series representation

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Similarly, for $\cos z$, $\sin z$

$$\cos z = 1 - z^2 + z^4 - \cdots$$

Similarly, for $\cos z$, $\sin z$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!}$$

Note $e^{iz} = \cos z + i \sin z$ (can be verified from these def.)

Then $\forall x \in \mathbb{R}$, $|e^{ix}| = \sqrt{\cos^2 x + \sin^2 x} = 1$.

$$\arg(e^{ix}) = \arg(\cos x + i \sin x) = \arctan \frac{\sin x}{\cos x} = \arctan \tan x = x.$$

Normal properties of exponentials also hold

$$e^0 = 1$$

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$$

We can also show

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Let's also define hyperbolic sine, cos, tan by

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$

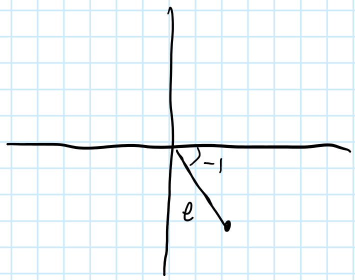
$$\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

Ex. $e^{1-i} = e^1 \cdot e^{-i}$

$$|e^{1-i}| = |e^1||e^{-i}| = e \cdot 1 = e$$

$$\arg\left(e^{1-i}\right) = \arg\left(e^{1}\cdot e^{-i}\right) = \arg\left(e^{1}\right) + \arg\left(e^{-i}\right) = 0 - 1 = -1$$



$$e^{1-i} = \underbrace{e}_{\text{modulus}} \cdot \underbrace{e^{-i}}_{\text{rotation}}$$

$x(1) = 0$

$x = 0$

$t = 1$

$$Ce^{x} + t^{2} - C\sin(t) = 0$$

$$C + 1 - C\sin(1) = 0$$

$$C - C\sin(1) = -1$$

$$C(1 - \sin(1)) = -1$$

$$C = \frac{-1}{1 - \sin 1}$$

$$x\ddot{x} + x\dot{x} + x = 0 \qquad x = 0$$

$$\ddot{x} + \dot{x} + 1 = 0$$